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A remark on a theorem of Chatterjee and last passage percolation

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Abstract

In this paper we prove universality of random matrix fluctuations of the last passage time of last passage percolation (LPP) in thin rectangles. The proof is a simple corollary of a theorem of Chatterjee. This gives an alternative (and more elementary) proof of the same result in Baik and Suidan (2005 *Int. Math. Res. Not.* 325–37) and Bodineau and Martin (2005 *Electron. Commun. Probab.* **10** 105–12 (electronic)) in the special case of finite third moment.

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1. Introduction

Random matrix theory (RMT) has been an active area of research ever since the pioneering papers of Wigner [23] and Dyson [11]. However, an explosion of research during the past decade has revealed that fundamental probability distributions first encountered in the fluctuation theory of RMT are in fact far more prevalent than at first expected. In particular, these distributions arise in areas as diverse as number theory, combinatorics, representation theory, statistics, graph theory and probability (see, for example, [1, 15, 22, 9, 17]).

One particularly striking example in which random matrix theory plays a role is in the problem of last passage percolation. Consider the $\mathbb{N} \times \mathbb{N}$ lattice and a family of associated independent identically distributed random variables $\{X_i^j\}_{i,j=1}^{\infty}$. An *up/right path* π from the site $(1, 1)$ to the site (N, k) is a collection of sites $\{(i_k, j_k)\}_{k=1}^{N+k-1}$ satisfying $(i_1, j_1) = (1, 1)$, $(i_{N+k-1}, j_{N+k-1}) = (N, k)$ and $(i_{k+1}, j_{k+1}) - (i_k, j_k) \in \{(1, 0), (0, 1)\}$. Let $(1, 1) \nearrow (N, k)$ denote the set of such up/right paths. The directed first and last passage times to $(N, k) \in \mathbb{N} \times \mathbb{N}$, denoted by $L^f(N, k)$ and $L^l(N, k)$, respectively, are defined by

$$L^f(N,k) = \min_{\pi \in (1,1) \nearrow (N,k)} \sum_{(i,j)\in\pi} X_i^j, \tag{1}$$

$$L^l(N,k) = \max_{\pi \in (1,1) \nearrow (N,k)} \sum_{(i,j)\in\pi} X_i^j. \tag{2}$$

If X_i^j is interpreted as the time to pass the site (i, j), $L^f(N, k)$ and $L^l(N, k)$ represent the minimal and maximal times, respectively, to travel from the site $(1, 1)$ to (N, k) along an admissible path. Since the directed last passage percolation time can be viewed as the departure time in queuing theory (see e.g. [12]), the discussion below applies to queuing theory. The discussion below also applies to the flux of particles at a given site in the totally asymmetric simple exclusion process (see e.g. [20]).

Johansson [14] discovered that if $\{X_i^j\}_{i,j=1}^{\infty}$ are independent identically distributed geometric random variables with parameter q, then the probability distribution for the correctly normalized fluctuations of the last passage time, L^l, is given by the Tracy–Widom Gaussian unitary ensemble (GUE) distribution function, F_{GUE} [21]. The precise statement of this remarkable fact is as follows. For any $\rho \in (0, 1]$,

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{L^l(N, \lfloor\rho N\rfloor) - c_1(\rho, q)N}{c_2(\rho, q)N^{\frac{1}{3}}} \leqslant s\right) = F_{\mathrm{GUE}}(s), \tag{3}$$

where c_1 and c_2 are explicit constants (depending only on ρ and q) and F_{GUE} is given by

$$F_{\mathrm{GUE}}(x) = \exp\left\{-\int_x^{\infty} (s - x)q^2(s)\,\mathrm{d}s\right\}, \tag{4}$$

where $q(x)$ solves the Painlevé II equation,

$$q'' = 2q^3 + xq \tag{5}$$

subject to the condition that $q(x) \sim \mathrm{Ai}(x)$ as $x \to +\infty$; $\mathrm{Ai}(x)$ denotes the Airy function. The function F_{GUE} is the limiting distribution function for the largest eigenvalue of a matrix chosen from the GUE as the dimension of the matrices grows to infinity (see [17, 21]). Johansson [14] also proved this result for exponential random variables. There are a few more models with different kinds of admissible paths that were proven to have the same limit law [2, 3, 13].

Johansson's proof relies heavily on the fact that $\{X_i^j\}_{i,j=1}^{\infty}$ are chosen to be geometric random variables; however, it is expected that a similar statement to (3) is true (for the fluctuations) for some fairly robust class of random variables. This remains a challenging open problem.

Recently, there have been several attempts to generalize Johansson's result [4, 6]. Both of these papers study the last passage problem for general random variables in thin rectangles (as opposed to the full scaling Johansson studies) by using certain natural functionals of Brownian motion and various strong approximation theorems which couple random walks to Brownian motion. The strong approximation/coupling results used in [4] and [6] are the Skorohod embedding theorem [10] and the Komlos, Major, Tusnady (KMT) theorem [16], respectively. The KMT theorem couples random walks whose increments have finite pth ($p > 2$) moment to Brownian motion while the Skorohod embedding theorem provides a coupling under the assumption $p = 4$. Both [4] and [6] obtain precisely the same result when $p = 4$. We record the general result as follows.

Theorem 1 ([6], see also [4] when $p = 4$)**.** *Suppose that $\{X_i^j\}_{i,j=1}^{\infty}$ is a family of independent identically distributed random variables such that $\mathbb{E}X_i^j = \mu$, $\mathbb{E}|X_i^j|^2 - \mu^2 = \sigma^2$ and $\mathbb{E}|X_i^j|^p < \infty$ for some $p > 2$. For any $s \in \mathbb{R}$,*

$$\lim_{N,k\to\infty} \mathbb{P}\left(\frac{L^l(N, k) - \mu(N + k - 1) - 2\sigma\sqrt{Nk}}{\sigma k^{-1/6}N^{\frac{1}{2}}} \leqslant s\right) = F_{\mathrm{GUE}}(s), \tag{6}$$

$$\lim_{N,k\to\infty} \mathbb{P}\left(\frac{L^f(N,k) - \mu(N+k-1) + 2\sigma\sqrt{Nk}}{\sigma k^{-1/6} N^{\frac{1}{2}}} \leqslant s\right) = 1 - F_{\mathrm{GUE}}(-s), \quad (7)$$

where $k = o(N^\alpha)$ *and* $\alpha < \frac{6}{7}\left(\frac{1}{2} - \frac{1}{p}\right)$.

The centring and normalization in theorem 1 can be explained in connection to random matrix theory, non-colliding random walks, and Brownian motion. The first term in the centring, $\mu(N+k-1)$, simply centres the random variables $\left\{X_i^j\right\}_{i,j=1}^{\infty}$ to have mean 0. The second term, $2\sigma\sqrt{Nk}$, is more interesting. If the random variables $\left\{X_i^j\right\}_{i,j=1}^{\infty}$ have mean 0, then as $N \to \infty$ while k remains fixed, the distribution of $N^{-\frac{1}{2}}L^l(N,k)$ converges to that of the top eigenvalue of a $k \times k$-GUE random matrix (see [5, 13, 19] for different perspectives on this fact). In order to arrive at the GUE Tracy–Widom limiting distribution for the top eigenvalue, $\lambda_1^{(k)}$, of a $k \times k$-GUE random matrix as $k \to \infty$, it is well known in random matrix theory that one scales as follows:

$$\lim_{k\to\infty} \mathbb{P}\left(k^{\frac{1}{6}}\left(\lambda_1^{(k)} - 2\sigma\sqrt{k}\right) \leqslant s\right) = F_{\mathrm{GUE}}(s). \quad (8)$$

This is exactly the scaling in theorem 1 with the important difference that $N, k \to \infty$ simultaneously.

It seems that when one uses either strong approximation/coupling theorem—Skorohod embedding or KMT—a technical problem arises at exactly the same α. This fact is not very surprising given that both [4] and [6] use essentially the same strategy.

The purpose of this paper is to explain a different proof of theorem 1 in the special case $\mathbb{E}\left|X_i^j\right|^3 < \infty$. Even though the proof which follows is completely different from the proof in [4, 6] it only applies for $\alpha < \frac{1}{7}$, the same seemingly technical restriction given by the original proof of theorem 1. It is somewhat surprising that both methods have exactly the same technical restriction. However, we remark here that it is possible that further restrictions must be imposed in order to prove Johansson's result for a general class of random variables. Indeed, as Martin pointed out to the author, Bernoulli (p) last passage percolation (in the full scaling limit) does not have RMT-type fluctuations if p is too close to 1.

2. Proof of theorem 1 for $\mathbb{E}\left|X_i^j\right|^3 < \infty$

The following proof is simply a corollary of recent theorems of Chatterjee [7] (see also [8]). Although Chatterjee's theorems have far reaching implications for universality, the proofs—which are inspired by Lindeberg's non-Fourier theoretic proof of the central limit theorem—are as technically elementary as they are elegant.

We state a combination of theorem 1.3 and corollary 1.2 of [7] in the most convenient form.

Theorem 2 (Chatterjee)**.** *Suppose that* $\mathbf{X} = (X_1, \ldots, X_n)$ *and* $\mathbf{Y} = (Y_1, \ldots, Y_n)$ *are two vectors of independent random variables taking values in* $\mathbb{R}$ *and satisfying, for each* i, $\mathbb{E}X_i = \mathbb{E}Y_i, \mathbb{E}X_i^2 = \mathbb{E}Y_i^2$ *and* $\gamma = \max\{\mathbb{E}|X_i|^3, \mathbb{E}|Y_i|^3, 1 \leqslant i \leqslant n\} < \infty$. *Suppose that* $\mathcal{F}$ *is a finite collection of coordinatewise thrice differentiable functions from* $\mathbb{R}^n$ *into* $\mathbb{R}$, $U = \max_{f\in\mathcal{F}} f(\mathbf{X})$, $V = \max_{f\in\mathcal{F}} f(\mathbf{Y})$, *and* g *is any thrice differentiable function. The following inequality holds:*

$$|\mathbb{E}g(U) - \mathbb{E}g(V)| \leqslant K(g)\left[(\gamma n \lambda_3(\mathcal{F}))^{\frac{1}{3}}(\log|\mathcal{F}|)^{\frac{2}{3}} + \gamma n \lambda_3(\mathcal{F})\right], \quad (9)$$

where $K(g) = \frac{19}{3}\|g'\|_\infty + 13\|g''\|_\infty + \frac{13}{3}\|g'''\|_\infty$ *and*

$$\lambda_3(\mathcal{F}) = \sup_{f\in\mathcal{F}}\left\{\sup\left\{\left|\partial_i^p f(x)\right|^{\frac{3}{p}} : 1 \leqslant i \leqslant n, 1 \leqslant p \leqslant 3, x \in \mathbb{R}^n\right\}\right\}. \tag{10}$$

In order to apply theorem 2 in our setting, let $\mathbf{X} = \left\{X_i^j\right\}_{i,j=1}^{N,k}$ and $\mathbf{Y} = \left\{Y_i^j\right\}_{i,j=1}^{N,k}$ be random vectors which satisfy the conditions of theorem 2. Let $f_\pi(x_1, \ldots, x_{Nk}) = \frac{k^{\frac{1}{6}}}{N^{\frac{1}{2}}}\left(\sum_{i_\ell\in\pi} x_{i_\ell} - \frac{\mu(N+k-1)+2\sigma\sqrt{Nk}}{\sigma}\right)$ for each $\pi \in (1,1) \nearrow (N,k)$ and $\mathcal{F} = \{f_\pi : \pi \in (1,1) \nearrow (N,k)\}$. One easily checks that $\lambda_3(\mathcal{F}) = \frac{k^{\frac{1}{2}}}{N^{\frac{3}{2}}}$. Using Stirling's approximation one can check that

$$|\mathcal{F}| = |(1,1) \nearrow (N,k)| = \frac{(N+k)!}{N!k!} \leqslant \left(\frac{N}{k}\right)^k \left(1 + \frac{k}{N}\right)^{N+k}. \tag{11}$$

Applying Chatterjee's theorem (theorem 2) leads to the following estimate:

$$|\mathbb{E}g(U) - \mathbb{E}g(V)| \leqslant K(g)\left[\left(\gamma Nk\frac{k^{\frac{1}{2}}}{N^{\frac{3}{2}}}\right)^{\frac{1}{3}}(2k(\log N - \log k) + 2(N+k)(\log(N+k) - \log N))^{\frac{2}{3}} + \gamma Nk\frac{k^{\frac{1}{2}}}{N^{\frac{3}{2}}}\right],$$

which vanishes as $N, k \to \infty$ if $k = o(N^\alpha)$ if $\alpha < \frac{1}{7}$ and $K(g) < \infty$.

Johansson [14] showed that if the random variables $\left\{X_i^j\right\}_{i,j=1}^\infty$ are geometric random variables, then $U \Longrightarrow F_{\mathrm{GUE}}$ in distribution. Chatterjee's theorem implies that the same is true for any correctly normalized and centred random vector $\mathbf{Y}$ (the means and second moments of the components of $\mathbf{Y}$ are centred and normalized to agree with those of $\mathbf{X}$) if $\alpha < \frac{1}{7}$. This completes the proof of theorem 1 in the case $\mathbb{E}\left|X_i^j\right|^3 < \infty$.

Although the proof presented here is based on different methods from those in [4, 6], the same technical condition is required. This leads to several natural questions.

(a) How large is the set of random variables that have the Tracy–Widom distribution as their limit under the procedure described? As noted at the end of the introduction, in the full scaling limit (i.e. $N \times \lfloor\rho N\rfloor$) the analogue of Johansson's theorem [14] is not even true unless some extra (and currently unknown) conditions are imposed on the random variables. For example, it seems that there should be no large point mass at the top of the support of the random variables [18].

(b) Alternatively, under only the condition of finite pth moment for the random variables $\{X_i^j\}_{i,j=1}^\infty$, how large can the rectangles be and still exhibit Tracy–Widom fluctuations?

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